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Products of quantum matrices

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Abstract. Given two quantum matrices $A \in GL_q(n)$ and $A' \in GL_{q'}(n)$ can one impose quadratic relations between elements of A and A' which (a) are consistent and (b) imply that the product matrix AA' is in $GL_{q''}(n)$? This question is studied systematically in two and three dimensions. In two dimensions, five families of solutions are found, one of which generalizes the relations between A and its inverse A^{-1}. For each family, the results are compared with the quantum transpose condition introduced by Ogievetsky and Wess. In three dimensions, we prove that non-trivial solutions are absent, a fact which prevents further generalization.

1. Introduction

This is an attempt to answer some questions regarding the product of quantum matrices in two and more dimensions. The $GL_q(n)$ quantum matrices considered in this paper may be viewed as linear mappings taking a quantum vector from the n-dimensional quantum plane to another quantum vector. Yet such mappings cannot be composed at will and produce a similar object. The square of a quantum matrix for example is not, in general, a quantum matrix itself. It is known, though, that it works in some situations. For example, the nth power of a $GL_q(2)$ matrix is a $GL_{q^n}(2)$ matrix [1, 2, 5], but this is not valid in higher dimensions. On the other hand, the product of two $GL_q(n)$ matrices whose elements commute with one another is again a $GL_q(n)$ matrix. This is no longer true if the parameters q are taken to be different or if the elements do not simply commute. For the two-dimensional case Ogievetsky and Wess [4] have unified these two situations by giving a sufficient condition for the product of two 2×2 quantum matrices to be itself a quantum matrix (presented in section 2). It involves the definition of a quantum analogue to the transposition of matrices and expresses as a set of relations among the elements of the two matrices containing quadratic monomials as well as inverted quantum determinants. These relations are, however, not sufficient to order any product of elements of both matrices. The existence of a set of quadratic relations that have this property and guarantee that the product is a quantum matrix should be of great help in order to generalize to higher dimensions and to shed some light on the product of quantum matrices. This paper focuses on finding such relations in two and three dimensions.

To achieve this a systematic procedure is applied. Taking two quantum matrices, one in $GL_q(n)$ the other in $GL_{q'}(n)$, we consider general quadratic relations among their elements and impose the fulfilment of the four requirements in turn. This method is described in section 3 using the $\hat{R}$-matrix formulation as well as illustrated on a special kind of relations valid in all dimensions and appearing later in this study. Section 4 is devoted to applying this procedure to the two-dimensional case. The quantum transpose condition is then shown to be satisfied *a posteriori*. It appears that the values of q and q' are very much restricted. In fact, we find that they can either be equal or inverse. Accordingly, the product lies in $GL_{q''}(n)$ only if q'' equals q, q' or their inverses. When $q' = \frac{1}{q}$ and $q'' = 1$ we have one more interesting solution which generalizes the construction of the quantum inverse matrix.

The same method is applied to 3×3 quantum matrices in section 5, bringing this time negative results, the only solution surviving the four requirements being the commuting one.

2. Quantum transpose condition

In this section we briefly present the q-transpose condition for the product of two 2×2 quantum matrices to be a quantum matrix. These results have been established for general two-parameter deformations of $GL(2)$, yet since we will consider only one-parameter deformations we will restrict this presentation to $GL_q(2)$.

2.1. The quantum transpose

Given a $GL_q(2)$ matrix T

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

with elements satisfying the usual $GL_q(2)$ relations

$$\begin{aligned} ab &= q\,ba \\ ac &= q\,ca \\ ad &= da + \lambda\,cb \\ bc &= cb \\ bd &= q\,db \\ cd &= q\,dc \end{aligned} \tag{2}$$

where $\lambda = q - q^{-1}$, its quantum transpose is defined as

$$^{\tau}T = \frac{1}{D}\begin{pmatrix} a & q\,c \\ b/q & d \end{pmatrix}. \tag{3}$$

D appearing in the above is the quantum determinant of T

$$D = \det_q T = da - \frac{1}{q}\,cb. \tag{4}$$

In the $GL_q(2)$ case D is central. Hence its inverse can be used easily, and put on either side of the matrix in the definition of the transpose.

2.2. q-transpose relations

We now consider two quantum matrices with different parameters: $T \in GL_q(2)$ and $T' \in GL_{q'}(2)$. T elements will be taken as before and T' will be primed, respectively. Their matrix product will be denoted by $T'' = T \cdot T'$,

$$T'' = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}. \tag{5}$$

The q-transpose condition reads as follows.

Theorem 1. *If there exists q'' and D'' such that*

$$
{}^{\tau}(T')\,{}^{\tau}(T) = \frac{1}{D''}\begin{pmatrix} aa' + bc' & q''\,(ca' + dc') \\ q''^{-1}\,(ab' + bd') & cb' + dd' \end{pmatrix} \tag{6}
$$

then $T'' \in GL_{q''}(2)$ and $D'' = \det_{q''}(T'')$.

When this condition is satisfied the right-hand side is precisely the quantum transpose of the quantum matrix T''. Thus, with some loss of rigour, the theorem can be stated as follows.

Corollary 1. *If ${}^{\tau}(T')\,{}^{\tau}(T) = {}^{\tau}(T \cdot T')$ then $T'' = T \cdot T' \in GL_{q''}(2)$.*

To see how such a condition can be satisfied, let us write the corresponding relations explicitly. We obtain four relations:

$$
\begin{aligned}
\frac{1}{D'}\left(a'a + \frac{q'}{q}\,c'b\right)\frac{1}{D} &= \frac{1}{D''}\,(aa' + bc') \\
\frac{1}{D'}\,(q\,a'c + q'\,c'd)\,\frac{1}{D} &= \frac{q''}{D''}\,(ca' + dc') \\
\frac{1}{D'}\left(\frac{1}{q'}\,b'a + \frac{1}{q}d'b\right)\frac{1}{D} &= \frac{1}{D''\,q''}\,(ab' + bd') \\
\frac{1}{D'}\left(\frac{q}{q'}\,b'c + d'd\right)\frac{1}{D} &= \frac{1}{D''}(cb' + dd').
\end{aligned} \tag{7}
$$

These relations do not allow for completing reordering of the products. To be able to order any product of T and T' elements one needs, in general, 16 quadratic relations. In the commuting case we have obvious ordering relations. The existence of such relations, in general, will be the main subject of the following sections.

3. General setting

3.1. Quantum matrices in n dimensions

Recall that the quantum group relations of $GL_q(n)$ can be expressed using an $\hat{R}$-matrix [3] as follows:

$$
\hat{R}^{ij}_{\alpha\beta}\,T^{\alpha}_{k}T^{\beta}_{l} = T^{i}_{\alpha}T^{j}_{\beta}\,\hat{R}^{\alpha\beta}_{kl} \tag{8}
$$

with

$$
\hat{R}^{ij}_{kl} = q^{\delta_{ij}}\,\delta^{i}_{l}\delta^{j}_{k} + \lambda\,\theta(l-k)\,\delta^{i}_{k}\delta^{j}_{l} \tag{9}
$$

where

$$
\begin{aligned}
&\lambda = q - q^{-1} \\
&\theta(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{otherwise} \end{cases} \\
&q^{\delta_{ij}} = 1 + (q-1)\,\delta_{ij}.
\end{aligned}
$$

Unless otherwise stated, in relation (8) and throughout this section summation will be performed over repeated Greek indices while Latin indices will never be summed. Note that relations (8) allow for complete ordering of products in lexicographic order:

$$
T^{i}_{j} < T^{k}_{l} \quad \Leftrightarrow \quad (i < j \text{ or } i = j, k < l). \tag{10}
$$

3.2. Requirements

So we will consider two quantum matrices $T \in GL_q(n)$ and $T' \in GL_{q'}(n)$ each satisfying the above relations with its own $\hat{R}$-matrix; $\hat{R}$ and $\hat{R}'$, respectively. Next we will impose four requirements:

(a) the relations must allow reordering of any product of T and T' elements;
(b) the product $T \cdot T'$ must be a quantum matrix with parameter q'';
(c) the relations must be compatible with the quantum group relations of both $GL_q(2)$ and $GL_{q'}(2)$;
(d) the relations must respect the $\mathbb{Z}$-grading of the matrices and their product.

In fact, an additional assumption to the first requirement will be used extensively in the subsequent calculations: the set of products $T^i_j T'^k_l$ is assumed to be linearly independent. Thus in the process of investigating requirement (b), the products $\{T^i_j T^k_l T'^s_t T'^u_v,\ T^i_j T^k_l$ ordered, $T'^s_t T'^u_v$ ordered$\}$ will be used as a basis for such products. Similarly, when studying requirement (c), the sets $\{T^i_j T^k_l T'^s_t,\ T^i_j T^k_l$ ordered$\}$ and $\{T^k_l T'^s_t T'^u_v,\ T'^s_t T'^u_v$ ordered$\}$ will be considered free. Though it eases the calculations and seems to stick to the most general situation of arbitrary T and T', this assumption is restrictive enough not to be satisfied in at least two known cases of a working product: the square in two dimensions, and the inverse.

General homogeneous, quadratic ordering relations among T and T' elements may be written as follows:

$$T'^i_j T^k_l = \Gamma^{ik\beta\varepsilon}_{jl\alpha\gamma}\, T^\alpha_\beta T'^\gamma_\varepsilon. \tag{11}$$

3.2.1. Grading. The grading is the usual $\mathbb{Z}$-grading of matrices. Let T be an $n \times n$ matrix, the grading of an element T^i_j is a sequence of length $k-1$ whose kth element is given by

$$\left[T^i_j\right] = G^i_j = \left(G^i_{j,1}, G^i_{j,2}, \ldots, G^i_{j,n}\right) \tag{12}$$

$$G^i_{j,k} = \theta(j-k) - \theta(i-k). \tag{13}$$

Thus in two dimensions we have

$$G = \begin{pmatrix} (0) & (1) \\ (-1) & (0) \end{pmatrix} \tag{14}$$

and in three dimensions

$$G = \begin{pmatrix} (0,0) & (0,1) & (1,1) \\ (0,-1) & (0,0) & (1,0) \\ (-1,-1) & (-1,0) & (0,0) \end{pmatrix}. \tag{15}$$

Both quantum group relations and the product respect this grading. The requirement that the crossing coefficients Γ respect it as well reads

$$\Gamma^{iktv}_{jlsu} \neq 0 \quad \Rightarrow \quad G^i_j + G^k_l = G^s_t + G^u_v. \tag{16}$$

The sum and equality on the right-hand side is taken term by term. Using (13) it translates to $\forall \xi \in \mathbb{N}_{n-1}$

$$\theta(j-\xi) + \theta(l-\xi) + \theta(s-\xi) + \theta(u-\xi) = \theta(i-\xi) + \theta(k-\xi) + \theta(t-\xi) + \theta(v-\xi). \tag{17}$$

We will not write the effect of this condition explicitly for the moment, though it will be taken into account later.

3.2.2. Product. We now come to the main requirement, that the product should be a quantum matrix i.e. $T'' = T \cdot T' \in GL_{q''}(n)$. In coordinates,

$$T''^i_{\ j} = T^i_\mu T'^\mu_{\ j} \tag{18}$$

and

$$\hat{R}''^{ij}_{\alpha\beta}\, T''^\alpha_{\ k} T''^\beta_{\ l} = T''^i_{\ \alpha} T''^j_{\ \beta}\, \hat{R}''^{\alpha\beta}_{kl} \tag{19}$$

or

$$\hat{R}''^{ij}_{\alpha\beta}\, T^\alpha_\sigma\, T'^\sigma_{\ k}\, T^\beta_\tau\, T'^\tau_{\ l} = T^i_\sigma\, T'^\sigma_{\ \alpha}\, T^j_\tau\, T'^\tau_{\ \beta}\, \hat{R}''^{\alpha\beta}_{kl}. \tag{20}$$

Applying the crossing relations (11) we obtain

$$\hat{R}''^{ij}_{\alpha\beta}\, \Gamma^{\sigma\beta\nu\psi}_{k\tau\mu\phi}\, T^\alpha_\sigma\, T^\mu_\nu\, T'^\phi_{\ \psi}\, T'^\tau_{\ l} = T^i_\sigma\, T^\mu_\nu\, T'^\phi_{\ \psi}\, T'^\tau_{\ \beta}\, \Gamma^{\sigma j\nu\psi}_{\alpha\tau\mu\phi}\, \hat{R}''^{\alpha\beta}_{kl}. \tag{21}$$

We then order the T–T and T'–T' products separately and identify the coefficients of the $TTT'T'$ monomials we obtain. We thus get a linear system of the Γ coefficients that we solve using *Mathematica*.

3.2.3. Compatibility. Checking the compatibility comes next. Formally, this is performed as follows. Let us take, for example, the expression

$$T'^a_{\ b}\left(\hat{R}^{ij}_{\alpha\beta}\, T^\alpha_k T^\beta_l - T^i_\alpha T^j_\beta\, \hat{R}^{\alpha\beta}_{kl}\right). \tag{22}$$

According to (8) this must be zero. Now let us move $T'^a_{\ b}$ through the parentheses using the crossing relations (11). We find something awful:

$$\hat{R}^{ij}_{\alpha\beta}\, \Gamma^{a\alpha\nu\chi}_{bk\mu\phi}\, \Gamma^{\phi\beta\rho\tau}_{\chi\rho\pi\sigma}\, T^\mu_\nu\, T^\pi_\rho\, T'^\sigma_{\ \tau} = T^\mu_\nu\, T^\pi_\rho\, T'^\sigma_{\ \tau}\, \Gamma^{ai\nu\chi}_{b\alpha\mu\phi}\, \Gamma^{\phi j\rho\tau}_{\chi\beta\pi\sigma}\, \hat{R}^{\alpha\beta}_{kl}. \tag{23}$$

After reordering and identifying the coefficients of the ordered monomials we obtain a system of quadratic equations in the coefficients Γ. These are half the equations that express compatibility. The other half is obtained by moving a T element through the primed version of relation (8) from the right. Even with the grading assumption this system is difficult to solve as is. That is why we solve it only after the linear system of the product condition has already restricted the coefficients.

3.3. A special case

There is one special case that will appear in the three-dimensional case and that easily generalizes to all dimensions. In that case, all q's are equal, namely

$$q = q' = q'' \tag{24}$$

$$\hat{R} = \hat{R}' = \hat{R}'' \tag{25}$$

the crossing tensor Γ has a very simple form:

$$\Gamma^{iktv}_{jlsu} = \tilde{\Gamma}^{it}_{ul}\, \delta^k_s \delta^v_j. \tag{26}$$

Substituting into (11) leads to

$$T'^i_{\ j}\, T^k_l = \tilde{\Gamma}^{i\beta}_{\alpha l} T^k_\beta T'^\alpha_{\ j}. \tag{27}$$

To comply with the grading, according to (27) and (13) we must have

$$G^i_j + G^k_l = G^k_\beta + G^\alpha_j \tag{28}$$

$\Leftrightarrow \forall \xi \in \mathbb{N}_{n-1}$

$$\theta(j-\xi)+\theta(l-\xi)+\theta(k-\xi)+\theta(\alpha-\xi) = \theta(i-\xi)+\theta(k-\xi)+\theta(\beta-\xi)+\theta(j-\xi) \tag{29}$$

$\Leftrightarrow \forall \xi \in \mathbb{N}_{n-1}$

$$\theta(l-\xi)+\theta(\alpha-\xi) = \theta(i-\xi+\theta(\beta-\xi) \tag{30}$$

$$\Leftrightarrow G^i_l = G^\alpha_\beta. \tag{31}$$

Hence $\tilde{\Gamma}^{ij}_{kl} \neq 0$ implies $G^i_l = G^j_k$, which can be achieved only if (i, l) and (j, k) represent the same position in the matrix, i.e. $i = l$, $j = k$, or if they are both diagonal, i.e. $i = j, k = l$. Thus $\tilde{\Gamma}$ must be of the form

$$\tilde{\Gamma}^{ij}_{kl} = \tilde{M}_{kl}\delta^i_k\delta^j_l + \tilde{N}_{kl}\delta^i_l\delta^j_k. \tag{32}$$

When $i = j$ and $k = l$, this gives $\left(\tilde{M}_{kk} + \tilde{N}_{kk}\right)\delta^i_k\delta_{kl}$ and allows us to set

$$\tilde{N}_{kk} = 0 \qquad \forall k \leqslant n \tag{33}$$

to normalize the diagonal.

3.3.1. Product. Let us see what the product condition gives. Substituting (26) into (21) we obtain

$$\hat{R}''^{ij}_{\alpha\beta}\,\tilde{\Gamma}^{\sigma\nu}_{\phi\tau}\,T^\alpha_\sigma\,T^\beta_\nu\,T'^\phi_k\,T'^\tau_l = T^i_\sigma\,T^j_\nu\,T'^\phi_\alpha\,T'^\tau_\beta\,\tilde{\Gamma}^{\sigma\nu}_{\phi\tau}\,\hat{R}''^{\alpha\beta}_{kl}. \tag{34}$$

Thanks to (25) we can use quantum group relations (8) on both sides:

$$\tilde{\Gamma}^{\sigma\nu}_{\phi\tau}\,\hat{R}^{ij}_{\alpha\beta}\,T^\alpha_\sigma\,T^\beta_\nu\,T'^\phi_k\,T'^\tau_l = T^i_\sigma\,T^j_\nu\,T'^\phi_\alpha\,T'^\tau_\beta\,\hat{R}'^{\alpha\beta}_{kl}\,\tilde{\Gamma}^{\sigma\nu}_{\phi\tau} \tag{35}$$

$$\tilde{\Gamma}^{\sigma\nu}_{\phi\tau}\,T^i_\alpha\,T^j_\beta\,\hat{R}^{\alpha\beta}_{\sigma\nu}\,T'^\phi_k\,T'^\tau_l = T^i_\sigma\,T^j_\nu\,\hat{R}'^{\phi\tau}_{\alpha\beta}\,T'^\alpha_k\,T'^\beta_l\,\tilde{\Gamma}^{\sigma\nu}_{\phi\tau}. \tag{36}$$

By renaming the right-hand side indices, dropping the unnecessary primes on $\hat{R}$'s and gathering terms, we finally obtain

$$T^i_\alpha\,T^j_\beta\left(\hat{R}^{\alpha\beta}_{\sigma\nu}\,\tilde{\Gamma}^{\sigma\nu}_{\phi\tau} - \tilde{\Gamma}^{\alpha\beta}_{\sigma\nu}\,\hat{R}^{\sigma\nu}_{\phi\tau}\right)T'^\phi_k\,T'^\tau_l = 0 \tag{37}$$

from which we deduce

$$\hat{R}^{ij}_{\alpha\beta}\,\tilde{\Gamma}^{\alpha\beta}_{kl} = \tilde{\Gamma}^{ij}_{\alpha\beta}\,\hat{R}^{\alpha\beta}_{kl}. \tag{38}$$

Thus in this case, $T'' \in GL_q(n)$ is equivalent to $\hat{R}\,\tilde{\Gamma} = \tilde{\Gamma}\,\hat{R}$.

We must now inject form (32) into this:

$$\tilde{M}_{kl}\,\hat{R}^{ij}_{kl} + \tilde{N}_{kl}\,\hat{R}^{ij}_{lk} = \tilde{M}_{ij}\,\hat{R}^{ij}_{kl} + \tilde{N}_{ji}\,\hat{R}^{ji}_{kl}. \tag{39}$$

Using the expression of the $\hat{R}$-matrix (9)

$$\delta^i_k\delta^j_l\,q^{\delta_{ij}}\left(\tilde{N}_{kl} - \tilde{N}_{lk}\right) + \delta^i_l\delta^j_k\left(q^{\delta_{ij}}\left(\tilde{M}_{kl} - \tilde{M}_{lk}\right) + \lambda\left(\theta(k-l) - \theta(l-k)\right)\tilde{N}_{kl}\right) = 0 \tag{40}$$

which finally gives

$$\tilde{N}_{kl} = \tilde{N}_{lk} \tag{41}$$

$$\tilde{M}_{kl} = \tilde{M}_{lk} + \lambda\ (\theta(k-l) - \theta(l-k))\,\tilde{N}_{kl}. \tag{42}$$

3.3.2. Compatibility. We then have to check for compatibility. It is expressed by equality (23). Substituting relations (27) into that equation, we obtain

$$\hat{R}^{ij}_{\alpha\beta}\, T^{\alpha}_{\nu}\, T^{\beta}_{\rho}\, \tilde{\Gamma}^{a\nu}_{\phi k}\, \tilde{\Gamma}^{\phi\rho}_{\sigma l}\, T'^{\sigma}_{\ b} = T^{i}_{\nu}\, T^{j}_{\rho}\, \tilde{\Gamma}^{a\nu}_{\phi\alpha}\, \tilde{\Gamma}^{\phi\rho}_{\sigma\beta}\, \hat{R}^{\alpha\beta}_{kl} T'^{\sigma}_{\ b}. \tag{43}$$

Applying (8) once more to the left-hand side and renaming indices on the right-hand side, leads to

$$T^{i}_{\alpha}\, T^{j}_{\beta}\left(\hat{R}^{\alpha\beta}_{\nu\rho}\, \tilde{\Gamma}^{a\nu}_{\phi k}\, \tilde{\Gamma}^{\phi\rho}_{\sigma l} - \tilde{\Gamma}^{a\alpha}_{\phi\nu}\, \tilde{\Gamma}^{\phi\beta}_{\sigma\rho}\, \hat{R}^{\nu\rho}_{kl}\right) T'^{\sigma}_{\ b} = 0 \tag{44}$$

and hence to

$$\hat{R}^{ij}_{\mu\nu}\, \tilde{\Gamma}^{a\mu}_{\phi k}\, \tilde{\Gamma}^{\phi\nu}_{bl} = \tilde{\Gamma}^{ai}_{\phi\mu}\, \tilde{\Gamma}^{\phi j}_{b\nu}\, \hat{R}^{\mu\nu}_{kl}. \tag{45}$$

Setting for each a and b,

$$(\Omega^{a}_{b})^{ij}_{kl} = \tilde{\Gamma}^{ai}_{\phi k}\, \tilde{\Gamma}^{\phi j}_{bl} \tag{46}$$

this simply reads

$$\hat{R}\, \Omega^{a}_{b} = \Omega^{a}_{b}\, \hat{R} \tag{47}$$

which is then equivalent to the following four relations:

$$i = j \quad k < l \qquad (\Omega^{a}_{b})^{ii}_{kl} = q\, (\Omega^{a}_{b})^{ii}_{lk} \tag{48}$$

$$i < j \quad k = l \qquad (\Omega^{a}_{b})^{ij}_{kk} = q\, (\Omega^{a}_{b})^{ji}_{kk} \tag{49}$$

$$i < j \quad k > l \qquad (\Omega^{a}_{b})^{ij}_{kl} = (\Omega^{a}_{b})^{ji}_{lk} \tag{50}$$

$$i < j \quad k < l \qquad (\Omega^{a}_{b})^{ij}_{kl} = (\Omega^{a}_{b})^{ji}_{lk} + \lambda\, (\Omega^{a}_{b})^{ji}_{kl}. \tag{51}$$

Using (33), definition (46) reads

$$(\Omega^{a}_{b})^{ij}_{kl} = \tilde{M}_{ak}\, \tilde{M}_{bl}\, \delta^{a}_{b}\, \delta^{i}_{k}\, \delta^{j}_{l} + \tilde{M}_{ak}\, \tilde{N}_{bl}\, \delta^{a}_{l}\, \delta^{i}_{k}\, \delta^{j}_{b} + \tilde{N}_{ik}\, \tilde{M}_{bl}\, \delta^{a}_{k}\, \delta^{i}_{b}\, \delta^{j}_{l}\, s + \tilde{N}_{ik}\, \tilde{N}_{bl}\, \delta^{a}_{k}\, \delta^{i}_{l}\, \delta^{j}_{b}. \tag{52}$$

Consider relation (48). The condition $k < l$ suppresses the first term in (52). Using the normalization of $\tilde{N}$ (33), it reads

$$\delta^{i}_{b}\left(\delta^{a}_{l}\, \delta^{i}_{k}\left(\tilde{M}_{lk}\, \tilde{N}_{kl} - q\, \tilde{N}_{kl}\, \tilde{M}_{kk}\right) + \delta^{a}_{k}\, \delta^{i}_{l}\left(\tilde{N}_{lk}\, \tilde{M}_{ll} - q\, \tilde{M}_{kl}\, \tilde{N}_{lk}\right)\right) = 0 \tag{53}$$

giving two relations for $k < l$

$$\tilde{N}_{kl}\big(\tilde{M}_{lk} - q\, \tilde{M}_{kk}\big) = 0 \tag{54}$$

$$\tilde{N}_{lk}\big(\tilde{M}_{ll} - q\, \tilde{M}_{kl}\big) = 0. \tag{55}$$

Relation (49) similarly gives for $i < j$

$$\tilde{N}_{ji}\big(\tilde{M}_{ii} - q\, \tilde{M}_{ji}\big) = 0 \tag{56}$$

$$\tilde{N}_{ij}\big(\tilde{M}_{ij} - q\, \tilde{M}_{jj}\big) = 0. \tag{57}$$

Renaming $i < j$ to $k < l$ and making use of $\tilde{N}_{kl} = \tilde{N}_{lk}$ (41) we have, for $k < l$,

$$\tilde{N}_{kl}\big(\tilde{M}_{lk} - q\, \tilde{M}_{kk}\big) = 0 \tag{58}$$

$$\tilde{N}_{kl}\big(\tilde{M}_{kk} - q\, \tilde{M}_{lk}\big) = 0 \tag{59}$$

and

$$\tilde{N}_{kl}\big(\tilde{M}_{ll} - q\, \tilde{M}_{kl}\big) = 0 \tag{60}$$

$$\tilde{N}_{kl}\big(\tilde{M}_{kl} - q\, \tilde{M}_{ll}\big) = 0. \tag{61}$$

which implies, for $k < l$,

$$\tilde{N}_{kl}\,\tilde{M}_{lk} = 0 \tag{62}$$

$$\tilde{N}_{kl}\,\tilde{M}_{kl} = 0. \tag{63}$$

Using (42), we can rewrite the second equation as

$$k < l \qquad \tilde{N}_{kl}\big(\tilde{M}_{lk} - \lambda\,\tilde{N}_{kl}\big) = 0. \tag{64}$$

The first equation then leads to

$$\begin{aligned} k < l \qquad & \tilde{N}_{kl}^2 = 0 \\ k < l \qquad & \tilde{N}_{kl} = 0. \end{aligned} \tag{65}$$

Equations (33), (41) and (65) simply mean that

$$\tilde{N} = 0. \tag{66}$$

It is easy to see that (48)–(51) are satisfied and bring no new restriction.

3.3.3. Summary. In the case when $q = q' = q''$, if we ask for crossing relations of the form (27)

$$T'^{i}_{\ j}\,T^{k}_{l} = \tilde{\Gamma}^{i\beta}_{\alpha l}\,T^{k}_{\beta}\,T'^{\alpha}_{\ j} \tag{67}$$

the grading, compatibility and product requirements constrain $\tilde{\Gamma}$ to

$$\tilde{\Gamma}^{ij}_{kl} = \tilde{M}_{kl}\,\delta^{i}_{k}\,\delta^{j}_{l} \tag{68}$$

where $\tilde{M}$ is a symmetric $n \times n$ matrix. We then simply have

$$T'^{i}_{\ j}\,T^{k}_{l} = \tilde{M}_{il}\,T^{k}_{l}\,T'^{i}_{\ j} \tag{69}$$

$$\tilde{M}_{il} = \tilde{M}_{li}. \tag{70}$$

4. Product of 2 × 2 matrices

In the two-dimensional case we have the results of section 2. Yet the presence of the determinants makes solving the transpose relations directly a difficult task. Indeed, as long as we do not have relations allowing one to move an unprimed element through a primed determinant and vice versa we have no easy way of manipulating the equations. We therefore chose to study the problem in the general setting of section 2 and check afterwards how these relations may extend the quantum transpose relations. Since the number of generators (4 + 4) and the number of coefficients (70) are acceptable, we will work directly with them without matrix and tensor indices.

4.1. Crossing relations

In two dimensions the grading is simply (14)

$$G = \begin{pmatrix} (0) & (1) \\ (-1) & (0) \end{pmatrix}. \tag{71}$$

The translation of relations (11) satisfying the grading conditions is as follows:

$$b'b = \mu_1\, bb' \tag{72}$$

$$a'b = \mu_2\, ba' + \mu_3\, ab' + \mu_4\, db' + \mu_5\, bd' \tag{73}$$

$$b'a = \mu_6\, ba' + \mu_7\, ab' + \mu_8\, db' + \mu_9\, bd' \tag{74}$$

$$b'd = \mu_{10}\, ba' + \mu_{11}\, ab' + \mu_{12}\, db' + \mu_{13}\, bd' \tag{75}$$

$$d'b = \mu_{14}\, ba' + \mu_{15}\, ab' + \mu_{16}\, db' + \mu_{17}\, bd' \tag{76}$$

$$a'a = \mu_{18}\, aa' + \mu_{19}\, da' + \mu_{20}\, cb' + \mu_{21}\, bc' + \mu_{22}\, ad' + \mu_{23}\, dd' \tag{77}$$

$$a'd = \mu_{24}\, aa' + \mu_{25}\, da' + \mu_{26}\, cb' + \mu_{27}\, bc' + \mu_{28}\, ad' + \mu_{29}\, dd' \tag{78}$$

$$b'c = \mu_{30}\, aa' + \mu_{31}\, da' + \mu_{32}\, cb' + \mu_{33}\, bc' + \mu_{34}\, ad' + \mu_{35}\, dd' \tag{79}$$

$$c'b = \mu_{36}\, aa' + \mu_{37}\, da' + \mu_{38}\, cb' + \mu_{39}\, bc' + \mu_{40}\, ad' + \mu_{41}\, dd' \tag{80}$$

$$d'a = \mu_{42}\, aa' + \mu_{43}\, da' + \mu_{44}\, cb' + \mu_{45}\, bc' + \mu_{46}\, ad' + \mu_{47}\, dd' \tag{81}$$

$$d'd = \mu_{48}\, aa' + \mu_{49}\, da' + \mu_{50}\, cb' + \mu_{51}\, bc' + \mu_{52}\, ad' + \mu_{53}\, dd' \tag{82}$$

$$a'c = \mu_{54}\, ca' + \mu_{55}\, ac' + \mu_{56}\, dc' + \mu_{57}\, cd' \tag{83}$$

$$c'a = \mu_{58}\, ca' + \mu_{59}\, ac' + \mu_{60}\, dc' + \mu_{61}\, cd' \tag{84}$$

$$c'd = \mu_{62}\, ca' + \mu_{63}\, ac' + \mu_{64}\, dc' + \mu_{65}\, cd' \tag{85}$$

$$d'c = \mu_{66}\, ca' + \mu_{67}\, ac' + \mu_{68}\, dc' + \mu_{69}\, cd' \tag{86}$$

$$c'c = \mu_{70}\, cc'. \tag{87}$$

The relations are presented from highest grading to lowest grading: the first equation contains only terms of grading (+2), the next four those of grading (+1) followed by the six relations of grading (0) and then the negative gradings. Obviously, not all the coefficients appearing in the same relation can be taken to be simultaneously zero. This simple remark will be of quite some use in the solving process.

The product requirement reads in components:

$$(aa' + bc')(ab' + bd') = q''\, (ab' + bd')(aa' + bc') \tag{88a}$$

$$(aa' + bc')(ca' + dc') = q''\, (ca' + dc')(aa' + bc') \tag{88b}$$

$$(aa' + bc')(cb' + dd') = (cb' + dd')(aa' + bc'') + \lambda''(ca' + dc')(ab' + bd') \tag{88c}$$

$$(aa' + bd')(ca' + dc') = (ca' + dc')(aa' + bd') \tag{88d}$$

$$(ab' + bd')(cb' + dd') = q''\, (cb' + dd')(ab' + bd') \tag{88e}$$

$$(ca' + dc')(cb' + dd') = q''\, (cb' + dd')(ca' + dc'). \tag{88f}$$

We expand these relations, apply the crossing relations to the central product in each term and then reorder the T–T and T'–T' product. We then identify the coefficients to obtain the system of linear equations constraining the μ_i's. After solving this system, the resulting crossing relations read

$$a'a = -q'\mu_3\, ad' - q'\mu_4\, dd' + \mu_{18}\, aa' + \mu_{19}\, da' + \mu_{21}\, bc' \tag{89}$$

$$a'b = \mu_2\, ba' + \mu_3\, ab' + \mu_4\, db' \tag{90}$$

$$a'c = -\frac{q'q''\mu_3}{q}\, cd' + \frac{q''\mu_{18}}{q}\, ca' + \left(q''\mu_{12} - q'\mu_{25} + \frac{q''\mu_{36}}{q'}\right) dc' \tag{91}$$

$$a'd = \frac{q''\mu_3}{q}\, cb' + \mu_{25}\, da' + \mu_{27}\, bc' \tag{92}$$

$$b'a = \frac{q'\mu_{18}}{q''}\, ab' + \frac{q'\mu_{19}}{q''}\, db' + \left(\frac{\mu_2}{q''} - \frac{\mu_{25}}{q} + \frac{q''\mu_{36}}{q}\right) bd' \tag{93}$$

$$b'b = \left(\frac{\lambda'}{q''}\mu_2 + \frac{\mu_{21}}{q''} + \frac{\mu_{25}}{qq'} - \frac{q''\,\mu_{36}}{qq'}\right) bb' \tag{94}$$

$$b'c = -q'\mu_{10}\, bc' + \frac{q'\mu_{18}}{q}\, cb' + \mu_{36}\, dd' \tag{95}$$

$$b'd = \mu_{10}\, ba' + \mu_{12}\, db' + \frac{q'\mu_{27}}{q''}\, bd' \tag{96}$$

$$c'a = qq'\mu_4\, cd' - q\mu_{19}\, ca' + \left(\mu_{12} - \lambda'\mu_{36}\right)\, ac' \tag{97}$$

$$c'b = -q\mu_4\, cb' + \mu_{36}\, aa' + \mu_{39}\, bc' \tag{98}$$

$$c'c = q''\left(-\lambda\mu_{12} + \mu_{21} + qq'\mu_{25} + \left(\frac{1}{qq'} - \frac{q}{q'} - \frac{q'}{q}\right)\mu_{36}\right) cc' \tag{99}$$

$$c'd = -q\mu_{27}\, ac' + (q''\mu_2 - q\mu_{25} + \frac{q''\mu_{36}}{q})\, ca' + \frac{q''\mu_{39}}{q}\, dc' \tag{100}$$

$$d'a = -\frac{qq'\mu_{19}}{q''}\, cb' + \left(\mu_{25} - \lambda''\mu_{36}\right)\, ad' - \frac{qq'\mu_{49}}{q''}\, bc' \tag{101}$$

$$d'b = \left(\frac{\mu_{12}}{q''} - \frac{\mu_{25}}{q'} + \frac{q''\mu_{36}}{q'}\right) ab' + \frac{q'\mu_{39}}{q''}\, bd' + \frac{q\mu_{49}}{q''}\, ba' \tag{102}$$

$$d'c = qq'\mu_{10}\, ac' + (\mu_2 - \lambda\mu_{36})\; cd' - q'\mu_{49}\, dc' \tag{103}$$

$$d'd = -q\mu_{10}\, aa' + \left(\lambda'\mu_2 - \lambda\mu_{12} + \mu_{21}\right)\, cb' - \frac{qq'\mu_{27}}{q''}\, ad' + \frac{q'\mu_{39}}{q}\, dd' + \mu_{49}\, da'. \tag{104}$$

4.2. Compatibility

We now impose compatibility conditions on the coefficients. After solving the system of quadratic equations in the μ_i's generated by these relations, it appears that the values of q' are restricted to $\{q, q^{-1}\}$ and those of q'' to $\{1, q, q^{-1}\}$, respectively. In fact, not all the combinations are allowed and we have five families of solutions. Before listing them let us make some remarks on the symmetries of the equation sets we have.

$GL_q(2)$ relations (2) are invariant under the following two transformations and their compose:

$$b \leftrightarrow c$$

and

$$a \leftrightarrow d$$
$$q \rightarrow q^{-1}.$$

Relations (88) expressing the product requirement are invariant under both compositions of these transformations for T and T':

$$\begin{array}{ll} a \leftrightarrow d & a' \leftrightarrow d' \\ b \leftrightarrow c & b' \leftrightarrow c' \\ q \rightarrow q^{-1} & q' \rightarrow q'^{-1}. \end{array} \tag{105}$$

This transformation will be denoted by Σ. The compatibility relations are also invariant under this symmetry.

Moreover, since both product and compatibility relations are homogeneous in both primed and unprimed elements, any overall scale factor introduced in relations (72)–(87) will become factored out. Thus we have the freedom of fixing one of the μ_i's conveniently. We will usually set

$$\mu_1 = 1. \tag{106}$$

The first family is diagonal and is exactly the result of section 1:

$$T'^i{}_j\, T^k_l = M^i_l\, T^k_l\, T'^i{}_j \tag{107}$$

with

$$M^i_l = M^l_i. \tag{108}$$

The next two families are for $q' = q^{-1}$, $q'' = 1$. The first reads

$$a'a = aa' + q^{-1}\lambda\, bc' \tag{109a}$$

$$a'b = q^{-1}ba' \tag{109b}$$

$$a'c = q^{-1}ca' \tag{109c}$$

$$a'd = da' \tag{109d}$$

$$b'a = q^{-1}\, ab' \tag{109e}$$

$$b'b = bb' \tag{109f}$$

$$b'c = q^{-2}\, cb' - q^{-1}\phi\, bc' \tag{109g}$$

$$b'd = q^{-1}\, db' + \phi\, ba' \tag{109h}$$

$$c'a = q^{-1}\, ca' \tag{109i}$$

$$c'b = bc' \tag{109j}$$

$$c'c = cc' \tag{109k}$$

$$c'd = q^{-1}\, dc' - \lambda\, ca' \tag{109l}$$

$$d'a = ad' \tag{109m}$$

$$d'b = q^{-1}\, db' - \lambda\, ab' \tag{109n}$$

$$d'c = q^{-1}\, cd' + \phi\, ac' \tag{109o}$$

$$d'd = q^{-1}\, dd' - q^{-1}\lambda\, cb' - q\phi\, aa'. \tag{109p}$$

And the second one is obtained by applying Σ (105).

The following two cases also occur for $q' = q^{-1}$ but this time we have either $q'' = q$ and

$$a'a = q\, aa' + \lambda\, bc' \tag{110a}$$

$$a'b = ba' \tag{110b}$$

$$a'c = q\, ca' + \lambda\, dc' \tag{110c}$$

$$a'd = da' \tag{110d}$$

$$b'a = q^{-1}\, ab' \tag{110e}$$

$$b'b = bb' \tag{110f}$$

$$b'c = q^{-1}\, cb' \tag{110g}$$

$$b'd = db' \tag{110h}$$

$$c'a = ac' \tag{110i}$$

$$c'b = q\, bc' \tag{110j}$$

$$c'c = cc' \tag{110k}$$

$$c'd = q\, dc' \tag{110l}$$

$$d'a = ad' \tag{110m}$$

$$d'b = q^{-1}\, bd' - \lambda\, ab' \tag{110n}$$

$$d'c = cd' \tag{110o}$$

$$d'd = q^{-1}\, dd' - \lambda\, cb' \tag{110p}$$

or $q'' = q^{-1}$ and

$$a'a = q\, aa' + \lambda\, bc' \tag{111a}$$

$$a'b = ba' \tag{111b}$$

$$a'c = q^{-1}\, ca' \tag{111c}$$

$$a'd = da' \tag{111d}$$

$$b'a = q\, ab' + \lambda\, bd' \tag{111e}$$

$$b'b = bb' \tag{111f}$$

$$b'c = q^{-1}\, cb' \tag{111g}$$

$$b'd = db' \tag{111h}$$

$$c'a = ac' \tag{111i}$$

$$c'b = q\, bc' \tag{111j}$$

$$c'c = cc' \tag{111k}$$

$$c'd = q^{-1}\, dc' - \lambda\, ca' \tag{111l}$$

$$d'a = ad' \tag{111m}$$

$$d'b = q\, bd' \tag{111n}$$

$$d'c = cd' \tag{111o}$$

$$d'd = q^{-1}\, dd' - \lambda\, cb'. \tag{111p}$$

The symmetry transformation Σ sends each of these solutions to itself.

4.3. Transpose conditions

Following our programme, we must finally check whether these solutions verify the quantum transpose relations of section 2. For the diagonal solutions this is obvious. We will therefore concentrate on the other four cases with $q' = q^{-1}$.

In order to compute the left-hand side of relations (7) we have to be able to express the commutation relations of D^{-1} with all T' elements. We will derive such relations from the corresponding commutations of D with T' elements. Now that we have full ordering relations for T–T' products it is possible to compute such relations.

4.3.1. $q' = q^{-1}$, $q'' = 1$ cases. Since these two cases exchange through Σ all the calculations are similar and we shall study only the first one, i.e. (109*a*)–(109*p*). We can write the commutations of D with T' elements as follows:

$$a'D = q^{-2}\, Da' + q^{-1}\lambda\, da'' \tag{112a}$$

$$b'D = q^{-2}\, Db' + \phi\, ba'' \tag{112b}$$

$$c'D = q^{-2}\, Dc' - \lambda\, ca'' \tag{112c}$$

$$d'D = q^{-2}\, Dd' - q\phi\, aa'' \tag{112d}$$

where $a'' = aa' + bc'$ is T''^1_1. This indeed allows us to invert D because we have the additional relations

$$[T^i_j, a''] = 0 \tag{113}$$

and consequently

$$[D, a''] = 0. \tag{114}$$

Thus, we can write

$$a'D^{-1} = q^2\, D^{-1}a' - q\lambda\, D^{-2}da'' \tag{115a}$$

$$b'D^{-1} = q^2\, D^{-1}b' - q^2\phi\, D^{-2}ba'' \tag{115b}$$

$$c'D^{-1} = q^2\, D^{-1}c' + \lambda\, D^{-2}ca'' \tag{115c}$$

$$d'D^{-1} = q^2\, D^{-1}d' + q^3\phi\, D^{-2}aa''. \tag{115d}$$

Substituting these in the left-hand side of equations (7) with $q' = q^{-1}$, we find

$$D'^{-1}(a'a + q^{-2}c'b)D^{-1} = D'^{-1}D^{-1}\left(q^2a'a + c'b - q\lambda D^{-1}(da - q^{-1}cb)a''\right) \tag{116a}$$

$$D'^{-1}(qa'c + q^{-1}c'd)D^{-1} = D'^{-1}D^{-1}\left(q^3a'c + qc'd - q^2\lambda D^{-1}(dc - q^{-1}cd)a''\right) \tag{116b}$$

$$D'^{-1}(qb'a + q^{-1}d'b)D^{-1} = D'^{-1}D^{-1}\left(q^3b'a + qd'b - \phi q^3 D^{-1}(ba - q^{-1}ab)a''\right) \tag{116c}$$

$$D'^{-1}(q^2b'c + d'd)D^{-1} = D'^{-1}D^{-1}\left(q^4b'c + q^2d'd - \phi q^3 D^{-1}(q\,bc - ad)a''\right). \tag{116d}$$

In the first and last relation, we recognize the q-determinant $D = ad - q\,bc = da - q^{-1}\,cb$ which we can simplify with D^{-1}. In the two central equations the last term simply cancels due to $GL_q(2)$ relations. We next apply the crossing relations (109*a*)–(109*p*) on the right-hand side and substitute $a'' = aa' + bc'$:

$$D'^{-1}D^{-1}\left(q^2\,(aa' + bc') - q\lambda\,(aa' + bc')\right) \tag{117a}$$

$$D'^{-1}D^{-1}\left(ca' + dc'\right) \tag{117b}$$

$$D'^{-1}D^{-1}\left(ab' + bd'\right) \tag{117c}$$

$$D'^{-1}D^{-1}\left(cb' + dd' - q^3\phi bc' - q^3\phi aa' + \phi q^3\,(aa' + bc')\right) \tag{117d}$$

which is precisely relations (7) with $q'' = 1$ and $D'' = DD'$. This last equality is easy to verify.

Before switching to the other cases let us make one remark. As one would expect from the values of q' and q'', these crossing relations are indeed related to the inverse of the quantum matrix T:

$$T^{-1} = D^{-1} \begin{pmatrix} d & -q^{-1} b \\ -q\, c & a \end{pmatrix}. \tag{118}$$

In fact, if we set

$$a' = d \qquad b' = -q^{-1} b \tag{119}$$

$$c' = -q\, c \qquad d' = a \tag{120}$$

the crossing relations are equivalent to the $GL_q(2)$ relations for T provided we also set

$$\phi = -q^2 \lambda. \tag{121}$$

Hence they generalize the relations between T and T^{-1}.

4.3.2. $q' = q^{-1}$, $q'' = q$ case. This case appears to be much simpler. Indeed, we have, for the determinant,

$$a' D = q\, D a' \tag{122a}$$

$$b' D = q^{-1}\, D b' \tag{122b}$$

$$c' D = q\, D c' \tag{122c}$$

$$d' D = q^{-1}\, D d \tag{122d}$$

hence

$$a' D^{-1} = q^{-1}\, D^{-1} a' \tag{123a}$$

$$b' D^{-1} = q\, D^{-1} b' \tag{123b}$$

$$c' D^{-1} = q^{-1}\, D^{-1} c' \tag{123c}$$

$$d' D^{-1} = q\, D^{-1} d. \tag{123d}$$

We also easily obtain $D'' = DD'$. Then the left-hand side of the q-transpose relations (7) reads

$$D'^{-1} (a'a + q^{-2} c'b)\, D^{-1} = D'^{-1} D^{-1} (q^{-1} a'a + q^{-3} c'b) \tag{124a}$$

$$D'^{-1} (q\, a'c + q^{-1} c'd)\, D^{-1} = D'^{-1} D^{-1} (a'c + q^{-2} c'd) \tag{124b}$$

$$D'^{-1} (q\, b'a + q^{-1} d'b)\, D^{-1} = D'^{-1} D^{-1} (q^2 b'a + d'b) \tag{124c}$$

$$D'^{-1} (q^2 b'c + d'd)\, D^{-1} = D'^{-1} D^{-1} (q^3 b'c + q\, d'd). \tag{124d}$$

Replacing the crossing relations (110a)–(110p) into these is straightforward and leads to relations (7).

4.3.3. $q' = q^{-1}$, $q'' = q^{-1}$ case. This case is less easy than the previous one. We have

$$a'D = q^{-1}\,Da' + \lambda\,da'' \tag{125a}$$

$$b'D = q^{-1}\,Db' + \lambda\,db'' \tag{125b}$$

$$c'D = q^{-1}\,Dc' - \lambda\,ac'' \tag{125c}$$

$$d'D = q^{-1}\,Dd' - \lambda\,ad''. \tag{125d}$$

Then we can summarize the relations between T'' and T elements by the following formula:

$$T''^{i}_{\;j}\,T^{k}_{l} = q^{3-k-i}\,T^{k}_{l}\,T''^{i}_{\;j} \tag{126}$$

and obtain

$$a''D = q\,Da'' \tag{127a}$$

$$b''D = q\,Db'' \tag{127b}$$

$$c''D = q^{-1}\,Dc'' \tag{127c}$$

$$d''D = q^{-1}\,Dd''. \tag{127d}$$

We can now invert D:

$$D^{-1}a'' = q\,a''D^{-1} \tag{128a}$$

$$D^{-1}b'' = q\,b''D^{-1} \tag{128b}$$

$$D^{-1}c'' = q^{-1}\,c''D^{-1} \tag{128c}$$

$$D^{-1}d'' = q^{-1}\,d''D^{-1} \tag{128d}$$

and

$$a'D^{-1} = q\,D^{-1}a' - \lambda\,D^{-2}da'' \tag{129a}$$

$$b'D^{-1} = q\,D^{-1}b' - \lambda\,D^{-2}db'' \tag{129b}$$

$$c'D^{-1} = q^{-1}\,D^{-1}c' + \lambda\,D^{-2}ac'' \tag{129c}$$

$$d'D^{-1} = q^{-1}\,D^{-1}d' + \lambda\,D^{-2}ad''. \tag{129d}$$

We check once again that $D'' = DD''$. We then substitute everything back in the left-hand side of the q-transpose relations (7) to see after somewhat longer calculations that we obtain the right-hand side as expected.

5. Product of 3×3 matrices

This time we will work in coordinates, since the number of generators and especially coefficients really becomes bigger than in the two-dimensional case. We will follow the steps of section 3 and take two quantum matrices $T \in GL_q(3)$ and $T' \in GL_{q'}(3)$ each satisfying relations (8) with its own $\hat{R}$-matrix; $\hat{R}$ and $\hat{R}'$, respectively. Once again T'' will be the matrix product of T and T' and the $\hat{R}$-matrix associated will be denoted by $\hat{R}''$.

The grading of 3×3 matrices is taken as in (15):

$$G = \begin{pmatrix} (0,0) & (0,1) & (1,1) \\ (0,-1) & (0,0) & (1,0) \\ (-1,-1) & (-1,0) & (0,0) \end{pmatrix}.$$

If we look at $T^i_j\, T'^k_l$ products we have one of grading (2, 2) and one of grading (−2, −2), two of each grading (2, 1), (1, 2), (1, −1), (−1, 1), (−2, −1), (−1, −2), one of each grading (2, 0), (0, 2), (−2, 0), (0, −2), eight of each grading (1, 1), (1, 0), (0, 1), (0, −1), (−1, 0), (−1, −1) and 15 of grading (0, 0) for a total of 81 bilinear monomials. Of course, it is exactly the same for $T'^i_j\, T^k_l$ products. This produces 639 coefficients for general graded ordering relations between T and T' elements.

Fortunately (?) the product condition restricts this number tremendously. In fact, after solving the equations corresponding to $T \cdot T'$ being a quantum matrix, we are left with two families of solutions: one is, as we shall see precisely of the kind studied in section 3.3. The second is of a similar form, though all q's are not equal.

5.1. *First family of solutions*

In this case, we have

$$q = q' = q'' \tag{130}$$

and

$$T'^i_j\, T^k_l = \left(\Phi^i_l - \lambda\, \Psi^i_l\, \theta(i-l)\right) T^k_l\, T'^i_j + \delta^i_l (1 - \delta^\alpha_l)\, \Psi^i_\alpha\, T^k_\alpha\, T'^\alpha_j \tag{131}$$

with

$$\Phi^i_j = \Phi^j_i \tag{132}$$

$$\Psi^i_j = \Psi^j_i \tag{133}$$

and, since they never appear anywhere,

$$\Psi^i_i = 0. \tag{134}$$

It is easy to see that this solution is of the form (27)

$$T'^i_j\, T^k_l = \tilde{\Gamma}^{i\beta}_{\alpha l} T^k_\beta T'^\alpha_j.$$

with

$$\tilde{\Gamma}^{ij}_{kl} = \left(\Phi^i_l - \lambda\, \Psi^i_l\, \theta(i-l)\right)\, \delta^i_k\, \delta^j_l + (1 - \delta_{kl})\, \Psi^i_k\, \delta^i_l\, \delta^j_k \tag{135}$$

and that it indeed satisfies $\hat{R}\, \tilde{\Gamma} = \tilde{\Gamma}\, \hat{R}$.

Hence according to the results of section 2.3.2, compatibility restricts this solution even further to

$$T'^i_j\, T^k_l = \Phi^i_l\, T^k_l\, T'^i_j \tag{136}$$

$$\Phi^i_l = \Phi^l_i. \tag{137}$$

5.2. *Second family of solutions*

It consists, in fact, of three families of solutions which are very similar if we name the q-parameters appropriately: setting

$$q'' = -\frac{1}{p} \tag{138}$$

we have the three following cases:

$$q = p \qquad q' = p^{-1} \tag{139}$$

$$q = p^{-1} \qquad q' = p \tag{140}$$

$$q = p \qquad q' = p \tag{141}$$

with

$$T'^i_{\ j}\, T^k_l = -\left(p\,\theta(i-l) + p^{-1}\,\theta(l-i)\right) \Psi^i_l\, T^k_l\, T'^i_{\ j} + \delta^i_l \left(1 - \delta^\alpha_l\right) \Psi^i_\alpha\, T^k_\alpha\, T'^\alpha_{\ j} \tag{142}$$

and as usual

$$\Psi^i_j = \Psi^j_i \qquad \Psi^i_i = 0. \tag{143}$$

We can then conduct similar calculations to those of section 3 to investigate compatibility. This unfortunately leads to similar results, i.e. $\Psi = 0$. Hence this family of solutions must simply be discarded.

5.3. Summary

In three dimensions, we are left with the single solution (136). This diagonal solution is, in fact, no more than the commuting case mentioned in the introduction with all the freedom on the diagonal coefficients expressed explicitly. Among these six coefficients, one can be set to one according to the same argument on an overall scale factor as in section 4. The other five express similar degrees of freedom in the quantum relations of each matrix and their product.

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References

[1] Corrigan E, Fairlie D B, Fletcher P and Sasaki R 1990 Some aspects of quantum groups and supergroups *J. Math. Phys.* **31** 776

[2] Ewen H, Ogievetsky O and Wess J 1991 Quantum matrices in two-dimensions *Lett. Math. Phys.* **22** 297–306

[3] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 Quantization of lie groups and lie algebras *Leningrad Math. J.* **1** 193–225

[4] Ogievetsky O and Wess J 1991 Relations between $GL_{p,q}(2)$s *Z. Phys.* C **50** 123–31

[5] Vokos S P, Zumino B and Wess J 1990 Analysis of the basic matrix representation of $gl(q)(2,\ c)$ *Z. Phys.* C **48** 65